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Variational formulation of time-dependent electromagnetic problems

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Abstract. Variational principles for electromagnetic fields in anisotropic inhomogeneous media for general time dependence are derived. The natural boundary conditions resulting from the formulations are investigated. By including proper boundary integrals, additional conditions may be satisfied. This leads to fewer restrictions on the trial fields and consequently a wider range of applicability.

1. Introduction

In recent years, there has been a growing interest in the analysis of time-dependent electromagnetic field problems. This interest is stimulated by various applications that require the explicit treatment of time-dependent effects. One such application is to study the transmission of signals through time-varying media as exemplified by the ionospheric or some other ionised plasma (Felsen 1976). For target identification problems, the transient or the impulse response is particularly useful since it contains electromagnetic information about the target beside being closely related to the target geometry (Moffat and Mains 1975). Also, short pulses of high power are finding application as diagnostic tools for the study of wave–material interaction. The role of transients in electromagnetic exploration was recently reviewed by Lee (1979).

There are basically two independent techniques available for solving transient electromagnetic problems. The usual approach to this subject followed the standard frequency domain technique which is subsequently transformed via Fourier series, Fourier or Laplace transforms to yield the desired time domain response. Alternatively, a direct formulation in the time domain may be used. Advantages of such a direct approach were discussed by Miller and Landt (1976).

Only a few electromagnetic transient solutions can be expressed in closed form in terms of standard functions. Consequently, most time domain solutions inevitably involve substantial computer processing or approximate techniques.

The variational technique is finding increasing applications in solving complex electromagnetic problems (Cairo and Kahan 1965, Mohsen 1978a, b, Chen and Lien 1980). The variational principle may summarise the equations concisely and include some or all the physical requirements as natural boundary conditions. The variational computation has the property that if the trial function assumed has a first-order error, the functionals computed from the variational expression have a second-order error. Thus, it provides a systematic way of making an optimum determination of successive

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approximations to the desired order of accuracy. Moreover, a very promising aspect of variational methods is the possibility of constructing various types of accuracy estimates of the functionals. The existence of upper and lower variational bounds on the functionals (Nikol'skiy and Feoktistov 1971, Kalikstein *et al* 1977) enables one, in principle, to approach the exact value monotonically. These bounds may be found with the help of complementary variational principles (Arthurs 1970).

The desired variational principle for time-dependent fields may be derived from the time harmonic form via application of the Fourier transform. However, a direct derivation in the time domain may be interesting and easier beside being applicable to fields whose time dependence is not Fourier transformable. It is well known that a general variational principle in classical physics may be based on the principle of least action (Morse and Feshbach 1953, Morishita and Kumagai 1977). Two variational expressions were derived by Welch (1960) and Cheo (1965) for scattering of time-dependent electromagnetic waves from a perfect conductor in free space. They employed Rumsey's reaction concept (Rumsey 1954) in their derivation. Using a canonical approach, Anderson and Arthurs (1979) presented several variational principles for the electromagnetic field vectors. By invoking the condition that the trial fields satisfy one set of Maxwell's equations or the remaining equations, complementary expressions were derived and were subsequently written in terms of auxiliary vector and scalar potentials.

Most of the previous studies were concerned with homogeneous isotropic media and did not give enough attention to the boundary conditions at interfaces. The present paper provides variational principles in terms of the field vectors as well as their associated Hertz vectors. The media considered are, in general, inhomogeneous and anisotropic. A study of the natural interface and boundary conditions is presented. Possible modifications to include additional conditions required in the formulation are considered.

2. Field equations

The medium under consideration is non-dispersive whose permittivity $\tilde{\varepsilon}$ and permeability $\tilde{\mu}$ are tensor functions of position, time independent and are assumed non-singular so that their reciprocals exist. The source excitations $\boldsymbol{J}$ and $\boldsymbol{M}$ are the electric and magnetic current densities, respectively. One associates electric and magnetic charge densities ρ and m with the above current densities via the continuity equations. The introduction of magnetic sources, beside being useful in some applications (Harrington 1961), puts the field equations in symmetrical form due to duality between electric and magnetic quantities. The sources are assumed to be located away from any surface of discontinuity.

If $\boldsymbol{E}$ and $\boldsymbol{H}$ denote the electric and magnetic field vectors, $\boldsymbol{D}$ and $\boldsymbol{B}$ the electric and magnetic flux densities, then Maxwell's equations read

$$\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}} - \boldsymbol{M} \qquad \nabla \cdot \boldsymbol{B} = m \tag{1}$$

$$\nabla \times \boldsymbol{H} = \dot{\boldsymbol{D}} + \boldsymbol{J} \qquad \nabla \cdot \boldsymbol{D} = \rho \tag{2}$$

where a dot over a function indicates derivative with respect to time. The constitutive relations are

$$\boldsymbol{D} = \tilde{\varepsilon}\boldsymbol{E} \qquad \boldsymbol{B} = \tilde{\mu}\boldsymbol{H}. \tag{3}$$

Using equations (1)–(3), the differential equations satisfied by $\boldsymbol{E}$ and $\boldsymbol{H}$ are given by

$$\nabla\times(\tilde{\mu}^{-1}\nabla\times\boldsymbol{E})+\tilde{\varepsilon}\ddot{\boldsymbol{E}}=-\boldsymbol{J}-\nabla\times(\tilde{\mu}^{-1}\boldsymbol{M}) \tag{4}$$

and

$$\nabla\times(\tilde{\varepsilon}^{-1}\nabla\times\boldsymbol{H})+\tilde{\mu}\ddot{\boldsymbol{H}}=-\boldsymbol{M}+\nabla\times(\tilde{\varepsilon}^{-1}\boldsymbol{J}). \tag{5}$$

Introducing the electric and magnetic Hertz vectors, it may be shown (Mohsen 1976) that both satisfy equations similar to (4) and (5) with appropriate source functions. Thus the problem of treating individual field or Hertz vectors reduces to the study of an equation in the form

$$\square(\tilde{p},\tilde{q})\boldsymbol{P}\equiv\nabla\times\boldsymbol{W}+\tilde{q}\ddot{\boldsymbol{P}}=\boldsymbol{Q} \tag{6}$$

where $\boldsymbol{W}=\tilde{p}\nabla\times\boldsymbol{P}$.

3. Derivation of the combined field variational principle

Since in the medium under consideration $\tilde{\mu}$ and $\tilde{\varepsilon}$ are non-symmetric tensors, a proper adjoint operator is defined in which these tensors are replaced by their transposes. The solution to this problem is derived using the original sources and this adjoint solution is denoted by $(\boldsymbol{E}^{\mathrm{a}},\boldsymbol{H}^{\mathrm{a}})$.

If the medium occupies a volume V bounded by a surface S and the time considered is between time $t=t_1$ and $t=t_2$, we define a volume inner product as

$$\langle\boldsymbol{P},\boldsymbol{Q}\rangle=\int_{t_1}^{t_2}\int_V\boldsymbol{P}\cdot\boldsymbol{Q}^{\mathrm{a}}\,\mathrm{d}V\,\mathrm{d}t. \tag{7}$$

It is also useful to introduce a surface product over a surface S as

$$(\boldsymbol{P},\boldsymbol{Q})_S=\int_{t_1}^{t_2}\int_S\boldsymbol{P}\cdot\boldsymbol{Q}^{\mathrm{a}}\,\mathrm{d}S\,\mathrm{d}t. \tag{8}$$

Following previous time-harmonic analysis (Mohsen 1978a), an appropriate functional may be written as

$$F=\langle\nabla\times\boldsymbol{E},\boldsymbol{H}\rangle-\langle\nabla\times\boldsymbol{H},\boldsymbol{E}\rangle+\langle\tilde{\mu}\dot{\boldsymbol{H}},\boldsymbol{H}\rangle+\langle\tilde{\varepsilon}\dot{\boldsymbol{E}},\boldsymbol{E}\rangle$$
$$+\langle\boldsymbol{M},\boldsymbol{H}\rangle-\langle\boldsymbol{H},\boldsymbol{M}\rangle+\langle\boldsymbol{J},\boldsymbol{E}\rangle-\langle\boldsymbol{E},\boldsymbol{J}\rangle. \tag{9}$$

To prove that equation (9) yields the required variational principle, one takes the first variation of F to obtain

$$\delta F=\langle\nabla\times\boldsymbol{E}+\tilde{\mu}\dot{\boldsymbol{H}}+\boldsymbol{M},\delta\boldsymbol{H}\rangle-\langle\nabla\times\boldsymbol{H}-\tilde{\varepsilon}\boldsymbol{E}-\boldsymbol{J},\delta\boldsymbol{E}\rangle-\langle\delta\boldsymbol{H},\nabla\times\boldsymbol{E}+\tilde{\mu}\dot{\boldsymbol{H}}+\boldsymbol{M}\rangle$$
$$+\langle\delta\boldsymbol{E},\nabla\times\boldsymbol{H}-\tilde{\varepsilon}\boldsymbol{E}-\boldsymbol{J}\rangle-(\delta\boldsymbol{E},\hat{\boldsymbol{n}}\times\boldsymbol{H})_S+(\delta\boldsymbol{H},\hat{\boldsymbol{n}}\times\boldsymbol{E})_S \tag{10}$$

where $\hat{\boldsymbol{n}}$ is a unit vector normal to S. In the process of derivation, use is made of variations of time derivatives of the form $\langle\delta\dot{\boldsymbol{A}},\boldsymbol{B}\rangle$ which is given by:

$$\langle\delta\dot{\boldsymbol{A}},\boldsymbol{B}\rangle=\int_{t_1}^{t_2}\int_V\frac{\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{A})\cdot\boldsymbol{B}^{\mathrm{a}}\,\mathrm{d}V\,\mathrm{d}t$$
$$=\int_V\delta\boldsymbol{A}\cdot\boldsymbol{B}^{\mathrm{a}}\,\mathrm{d}V\Big]_{t_1}^{t_2}-\int_{t_1}^{t_2}\int_V\delta\boldsymbol{A}\cdot\boldsymbol{B}^{\mathrm{a}}\,\mathrm{d}V\,\mathrm{d}t. \tag{11}$$

The first term in the integration by parts reduces to zero since the field is assumed fixed at the ends of the interval and we obtain

$$\langle \delta\dot{\boldsymbol{A}}, \boldsymbol{B}\rangle = -\langle \delta\boldsymbol{A}, \dot{\boldsymbol{B}}\rangle. \tag{12}$$

Also, Green's identity is used in the derivation. Consequently, at any surface of discontinuity surface integrals similar to those in (10) are to be added on both sides of the surface.

The stationarity of the functional given by (9) implies that the fields satisfy Maxwell's equations. The natural boundary conditions are the continuity of the tangential electric and magnetic fields at any surface of discontinuity, while these tangential fields vanish on the boundary. If the boundary is perfectly conducting, the addition of $(\boldsymbol{E}, \hat{\boldsymbol{n}} \times \boldsymbol{H})_S$ to equation (9) yields the required conditions, i.e. that only the tangential electric field vanishes.

Admissible trial fields must be continuous together with their first derivatives and must possess finite second derivatives everywhere except at surfaces where $\tilde{\varepsilon}$ and $\tilde{\mu}$ are discontinuous. At these surfaces, trial fields must satisfy the continuity of normal electric and magnetic flux densities (Berk 1956).

4. Derivation of variational principles in terms of one field vector

The formulation of a variational principle in terms of one field vector only, rather than both $\boldsymbol{E}$ and $\boldsymbol{H}$, has many computational advantages as pointed out by English and Young (1971). In the process of computation, it leads to the reduction of size of the solution matrices which also become denser. This reduces both the storage and time requirements and increases the stability of the computation.

As pointed out previously, the electric and magnetic field vectors as well as the Hertz vectors satisfy equation (6). Following previous time-harmonic analysis (Mohsen 1978b) an appropriate functional may be written as

$$F = \langle \boldsymbol{W}, \boldsymbol{\nabla} \times \boldsymbol{P}\rangle - \langle \tilde{q}\dot{\boldsymbol{P}}, \dot{\boldsymbol{P}}\rangle - \langle \boldsymbol{Q}, \boldsymbol{P}\rangle - \langle \boldsymbol{P}, \boldsymbol{Q}\rangle. \tag{13}$$

That (13) yields a variational principle under variations of $\boldsymbol{P}$ and $\boldsymbol{P}^{a}$ may be demonstrated by evaluating the first variation of F to obtain

$$\delta F = \langle \boldsymbol{\nabla} \times \boldsymbol{W} + \tilde{q}\ddot{\boldsymbol{P}} - \boldsymbol{Q}, \delta\boldsymbol{P}\rangle + \langle \delta\boldsymbol{P}, \boldsymbol{\nabla} \times \boldsymbol{W} + \tilde{q}\ddot{\boldsymbol{P}} - \boldsymbol{Q}\rangle$$
$$- (\hat{\boldsymbol{n}} \times \boldsymbol{W}, \delta\boldsymbol{P})_S - (\delta\boldsymbol{P}, \hat{\boldsymbol{n}} \times \boldsymbol{W})_S. \tag{14}$$

The stationarity of the expression implies that the field vector satisfies the appropriate equation given by equation (6). The natural boundary condition of the problem is the continuity of the tangential $\boldsymbol{W}$ at any surface of discontinuity while at the outer surface $\hat{\boldsymbol{n}} \times \boldsymbol{W} = 0$.

It is to be noted that the variational principle in terms of one field vector satisfies only one condition at the interface while using the combined field formulation satisfies two conditions. If the continuity of $\hat{\boldsymbol{n}} \times \boldsymbol{P}$ is to be implemented, surface integrals of the form $(\hat{\boldsymbol{n}} \times \boldsymbol{P}, \boldsymbol{P})_{S_i}$ at both sides of discontinuity at the surface S_i, are to be added to equation (13). It is worth noting that such integrals can be multiplied by constants which can be used as accelerating parameters. These parameters may be useful in accelerating the process of computation (Davies 1973).

If $\boldsymbol{P}$ stands for $\boldsymbol{H}$ and the external boundary is perfectly conducting, the disappearance of the tangential electric field is satisfied and there is no need to add a surface integral as in the combined field formulation. On the other hand, if $\boldsymbol{P} \equiv \boldsymbol{E}$, the tangential electric field vanishes upon the addition of $(\hat{\boldsymbol{n}} \times \boldsymbol{W}, \boldsymbol{P})_S + (\boldsymbol{P}, \hat{\boldsymbol{n}} \times \boldsymbol{W})_S$ to equation (13).

5. Conclusion

This paper studies the formulation of variational principles for time-dependent electromagnetic fields in anisotropic inhomogeneous linear media. In order to deal with the case when $\tilde{\mu}$ and $\tilde{\varepsilon}$ are non-symmetric tensors, a proper adjoint operator is introduced. The natural boundary conditions resulting from the formulation are discussed and means to satisfy additional required conditions are presented. This leads to increasing the range of applicability of the formulation and to the possibility of using simpler trial functions.

For a particular problem, the choice of a certain variational formula in preference to others largely depends on computational and physical considerations. For example, if the configuration of the electric field can be guessed more readily than that of the magnetic field, then it is sensible to use a formula in terms of the electric vector only. The advantages of using electric rather than magnetic vector formulation as far as convergence is concerned were discussed by English and Young (1971). The question of uniqueness of the solution was considered for the frequency domain formulation by Konrad (1976).

The present formulations encompass very general situations and simplifications result, naturally, upon considering particular cases. When time-harmonic variation is considered, the formulations are then useful in studying the propagation in waveguides, the cavity resonances, the estimation of scattering matrix elements, etc (Morishita and Kumagai 1977). The direct time-domain formulation may find a major application in the finite-element solutions of general time-varying electromagnetic problems. In such solutions, appropriate variational principles are needed in the construction of the procedure.

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